Nonperturbative effects in deformation quantization

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Abstract

The Cattaneo-Felder path integral form of the perturbative Kontsevich deformation quantization formula is used to explicitly demonstrate the existence of nonperturbative corrections to naïve deformation quantization.

The physical context of the formal problem of deformation quantization is the original one set out by Dirac [1] in making the substitution

$$\{f,g\} \to \frac{1}{i\hbar}\left(\hat{f}\star\hat{g}-\hat{g}\star\hat{f}\right) \tag{1}$$

the basis for relating classical mechanics on a phase space M to quantum mechanics, with $\hat{f},\hat{g}$ operators acting on a Hilbert space of wavefunctions.

In mathematical terms given a Poisson structure on a manifold M, the problem [2] is to find an associative product $\star$ on the space of formal power series in $\hbar$ with coefficients in the space of smooth functions on M such that

$$f\star g = fg + \frac{i\hbar}{2}\{f,g\} + O(\hbar^2) \tag{2}$$

where $\{f,g\}$ is the Poisson bracket on M. Kontsevich [3] gave a solution to this deformation problem in terms of a formal power series organized as a sum over graphs. The details of his construction will not be important for us.

What is more relevant for physics is the reformulation of his construction found by Cattaneo and Felder [4], who gave a path integral form of the Kontsevich formula. Recall now the Cattaneo-Felder construction: Let M be a Poisson manifold with a Poisson bracket given locally by

$$\{f,g\} = \sum_{i,j=1}^{d}\alpha^{ij}\partial_i f\partial_j g \tag{3}$$

with α a section of $\wedge^2 T_* M$ satisfying the Jacobi identity

$$\alpha^{il}\partial_l\alpha^{jk} + \text{cyclic } = 0. \tag{4}$$

Let X be a map from the unit disk $D^2 \to M$ and let η be a section of $X^*T^*M \otimes T^*D^2$. Then the action

$$S \equiv \int_{D^2} \eta_i \wedge \mathrm{d}X^i + \frac{1}{2}\alpha^{ij}\eta_i \wedge \eta_j \tag{5}$$

can be used to define a functional integral such that

$$f \star g(x) = \int_{X(1)=x} \mathrm{D}X\mathrm{D}\eta \exp(iS/\hbar) f(X(-1))g(X(i)) \tag{6}$$

where the integral is over all maps $X : D^2 \to M$ such that $X(1) = x$.

When the Poisson structure is associated with a symplectic form ω the Cattaneo-Felder formula simplifies considerably. The field η can be integrated out and one is left with

$$f \star g(x) = \int_{X(1)=x} \mathrm{D}X \exp\left(i \int_{X(\partial D^2)} \mathrm{d}^{-1}\omega/\hbar\right) f(X(-1))g(X(i)) \tag{7}$$

where the path integral is now over maps from the circle S^1 regarded as the boundary of the disk D^2 to the symplectic manifold M. Since ω is closed by definition, it can be represented locally as a one-form denoted symbolically as $\mathrm{d}^{-1}\omega$. In local Darboux coordinates the integral over the boundary is $S = \int p\mathrm{d}q$, therefore the equations of motion imply that the boundary value is locally constant. In other words, the classical equation of motion maps the boundary of D^2 to a point in M. This fact will be important for us in the following.

As in any aspect of quantum physics, the path integral is more fundamental than its perturbative saddle-point evaluation, so it is appropriate to investigate the Cattaneo-Felder path integral in detail to understand its physical content. I aim to demonstrate here that there are nonperturbative contributions to the Cattaneo-Felder path integral. These come from topologically nontrivial configurations, and hence have coefficients of the form $\exp(ic/\hbar)$. Since these contributions appear as essential singularities in a formal expansion in powers of $\hbar$, the nonperturbative deformation is still a solution to the formal deformation problem in eq. 2.

When will there be nontrivial solutions to the classical equations of motion? We wish to evaluate a sum over maps from the disk D^2 to the symplectic manifold M such that the boundary of the disk S^1 is mapped to the point x in M. In general, homotopy classes of maps from the n-disk D^n to a manifold M relative to a submanifold N of M which map the boundary of D^n to N are elements in the relative homotopy group $\pi_n(M, N)$. (Relative and absolute homotopy groups are defined with a choice of basepoint, but we have not made this dependence explicit in our notation.) In our case, we evidently need to consider $\pi_2(M, N \equiv \{x\})$, but this is isomorphic to the absolute homotopy group $\pi_2(M)$.

We expect then that we will get nonperturbative contributions to the path integral in cases where M has non-vanishing π_2. For example, $\pi_2(S^2) = \mathbf{Z}$ and $\pi_2(T^{2n}) = 0$ so there should be such contributions for S^2 and no such contributions for T^{2n}.

In the simplest case $M = S^2$, the homotopy classes of maps are just classified by the degree of the map. It is then easy to evaluate the action for degree n solutions to the classical equations of motion X_n since

$$\deg(X_n) \int_{S^2} \omega = \int_{D^2} X_n^*\omega \equiv S \tag{8}$$

and $V \equiv \int_{S^2} \omega$ is just the symplectic volume of S^2. Notice that the value of the action of X_n does not depend on the detailed form of X_n, just on the topological class given by the degree n.

Thus the Cattaneo-Felder path integral evaluated semiclassically for $M = S^2$ is

$$\sum_{n \in \mathbf{Z}} \exp(inV/\hbar) \langle f(X_n + \xi(-1)) g(X_n + \xi(i)) \rangle_n \tag{9}$$

where $\langle \ldots \rangle_n$ denotes the expectation value in the path integral evaluated perturbatively about X_n, with ξ the fluctuation. Thus we see contributions with essential singularities as functions of the deformation parameter $\hbar$ from topologically nontrivial sectors of the path integral.

I should add a few words on the perturbative evaluation of the path integral about these solutions. The analysis here is restricted to symplectic manifolds and one does not need all the sophistication necessary for the general case [4]. The gauge symmetry of the model in this case is diffeomorphism invariance since the action does not depend on any metric on the disk. If we consider Kähler manifolds, then a natural gauge fixing would be to localize on holomorphic representatives in each topological sector. This can be done in a straightforward fashion with a small modification of Witten's work on topological sigma models [5]. The standard formal path integral argument for associativity of the product is unchanged of course.

I want to emphasize that there is no shortcoming from a mathematical point of view in the work of Kontsevich [3]. His formula is perfectly adequate as a *formal* deformation quantization, but the importance of nonperturbative contributions cannot be over-emphasized from the physical perspective. Indeed, sums over such topologically nontrivial configurations are crucial for the appearance of mirror symmetry for example. There is no obvious duality symmetry that one expects in the simple example of $M = S^2$ considered here, but in general any T-duality like symmetry would require inclusion of such topologically non-trivial configurations in the path integral. π_2 is nontrivial for any simply connected Calabi-Yau manifold, for example.

I do not see immediately how to carry out this argument in the general case of a Poisson manifold (eq. 6) since the local constancy of the boundary classical configuration was crucial in identifying nontrivial configurations with elements in the homotopy group. The Seiberg-Witten [6] limit presumably requires the general case.

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